GEOMETRY IN THE 20TH CENTURY

M. Berger
IHÉS (Institut des Hautes Études Scientifiques), Bures sur Yvette, France

Keywords: connection, convex, curvature differential form, duality, dimension, fiber bundle, homogenous space, manifold, metric space, non-commutative, observable, point, polytope, projective space, Riemannian, spinor, space, symmetric space, sphere, trinity

Contents

Summary

This chapter not only describes the most important realizations of geometry (understood in its classical sense, in contrast to algebra and analysis) in the 20th century, but also the surprising fact that the geometric culture and vision had an enormous influence also in the other parts of mathematics—Analysis and Algebra.

1. Introduction

The reader who wants to go fast can get a good idea of the present topic by just looking at Section 2, which seems to us a good panorama. At the end we give a short annotated bibliography on our topic.

1.1. About History of Mathematics

Writing the history of mathematics over a long period of time and for such a broad subject is a difficult task. We now will spend, especially for the non-mathematicians, quite some time with this kind of considerations, since, in fact, the real nature of mathematics is, in general, extremely poorly perceived. That is why the philosophy of the present writing should be made precise. In our case we follow André Weil who says that what is interesting is not the formal aspect, say mathematics consists in a collection of axioms (or say concepts) and theorems, we can call the totality of them the *Hilbert tree.* Then a historical writing can consist in quoting all the relevant Hilbert's tree elements. In Art one can analogously just make just a catalogue of a given painter's works, or of a museum collection. Completely different is Gombrich "History of Art" where the aim is: to explain what was the desire moving a painter for one painting, or the same for an architect. Writing history of mathematics, and beside the desire of a given mathematician, one can hope to find inside Hilbert tree patterns of high density and, if elements of it are indexed, labeled with the time of proof, to find particularly strong currents, and if possible motivations for them. That is what André Weil desired, and what we will try to do here. This way of writing is exemplarily presented in Michael Atiyah Fields lecture "Mathematics in the 20th Century". Moreover we will freely use many of his insights when they are related to Geometry properly.

To argue differently, let us add some other thoughts of great mathematicians. In (Arnold 2000) one can find the following quotation of Sylvester, which is apparently in favor of: theorems, especially strong ones, summarizing many previous results "General statements are simpler than their particular cases", but beware of Sylvester next saying "A mathematical idea should not be petrified in a formalized axiomatic setting, but should be considered instead as flowing like a river". And now Arnold goes on "The experience of the past centuries shows that the development of mathematics was not due to technical progress (even if consuming most of the efforts of mathematicians at a given moment), but rather to discoveries of unexpected interrelations between different domains (which were made possible by these efforts). Let us quote André Weil again

:"…car le grand mathématicien de l'avenir, comme celui du passé, fuira les sentiers battus ; c'est par des rapprochements imprévus, auxquels notre imagination n'aura pas su atteindre, qu'il résoudra, en les faisantt changer de face, les grands problèmes que nous lui léguerons (because the great mathematician of the future, as the one of the past, will stray from the beaten track ; it is trough unexpected merging, which our imagination would not be able to attain, that he will solve, by forcing them to change their looking, the great problems that we will bequeath to him.)*".*

For example Arnold's text is written systematically for pointing these interrelations, Atiyah's one is more describing various important flows. We will try to mention the most important flows of the 20th century Mathematics which can be related essentially to Geometry.

Concerning those flows, it might be good to remark that some of them are discovered in some sense "afterwards" when looking at the whole Hilbert tree, some other are latent in the writings of great mathematicians, where they appear as "conjectures", or as "research programs". On the conjecture side, in Geometry, we can mention: Poincaré's conjecture (see the end of Section 2.5), the four color conjecture, Kepler conjecture. On the program side: Langlands philosophy, Gromov's various programs on Sub-Riemannian geometry (see Section 2.6) and on mm-spaces (see Section 2.8)**,** on negative curvature (Section 4.8), random groups, last but not the least Connes's program “Non commutative geometry”, see Section 2.14.

However, for limited space reasons, for competence reason and also for personal taste the author will not try to cover too systematically our subject. We apologize to the reader, as well as to our colleagues who are fond of Geometry, or those who contributed to the various fields of Geometry.

We will also point out some open problems which look quite simple, natural, but are still in active research.

1.2. The Quite Universal Domination of Geometry in the 20th Century Mathematics

It is time now to try to define what “Geometry” is. This is not easy; we have collected quite a lot of definitions interviewing great mathematicians. So we think it is simpler to be very brief, and still not too far from the reality by saying that:

Geometry is the study of the figures of/in space(s)

Let us now explain the surprising wording “universal”, since in general one is used to say that mathematics consists in Algebra, Analysis and Geometry. In Dieudonné and in Atiyah the geometry domination is detailed, and justified *de facto*. Of course, algebraic results are not geometric results, but both the language and the creation of such results by mathematicians are geometric in some sense. According to Dieudonné, it is in 1870 that the possibility of using a conventional language derived from the classical geometry, but of course without pretending that this was corresponding to an underlying physical reality. We also can quote Connes: “le difficile, l’essentiel en mathématiques, c’est de créer assez d’images mentales pour que le cerveau puisse fonctionner. Pour les algébristes, les images mentales et leur jeux, sont comparables à la syntaxe (the difficult, the essential in mathematics, is to create enough mental images in order that the brain can function. For the algebraists, their mental images and their games are comparable with syntax)”. The success of this “stealing” of geometric notions and wordings is enormous, of course not only in the language but also in the results.

Before coming back to Geometry proper, let us just mention a few typical topics. In Number theory (this is today the wording for arithmetic), Minkowski invented around 1900 a new field called “Geometric Number theory,” based on his geometric vision on convexity (see Section 7). Today there is a very recent new discipline of Number theory which is called “Arakelov geometry”, one justification being the use of arithmetic analogues of fiber bundles (see Section 2.9). There is also in Analysis the “Geometry of Banach spaces”, again invaded by the concept of Convexity (Section 7). When studying packing of balls, which is a subject very close to transmission of information, the notion of Hamming distance is basic. In recent theoretical physics “String theory” became fundamental. In the study of the universe, some theories considered that our universe is made of two different spaces which are extremely close, for relations between Geometry, Mathematics and Physics. The wording “Riemann surface” is in disguise a central topic in Analysis when one studies holomorphic functions of complex variable. Note however that the pure geometric study of Riemann surfaces came back recently, see Section 4.1. A trivial example of geometric wording is the word *hyperplane*, or *hypersurface*, to denote in an n-dimensional space (see Section 2.3) an object of dimension just one below.

Now we cannot escape our readers demand: how to explain the fact that the geometric vision of (some) mathematicians can help them to make important discoveries in many, if not all, fields which are completely conceptual (worse: successive accumulation of abstract concepts one above the other, on order to reach the sky, like in the Greek myth of Pelion over Ossa, or the Bible Jacob's ladder) For this Dieudonné is explaining it using the wording "transfer of intuition", but this is not too much illuminating. Atiyah goes deeper: "Vision uses up something like 80 or 90 percent of the cortex of the brain. They are about 17 different centers in the brain, each of which is specialized in a different part of the process of vision…". The complete text of Atiyah is worth reading, but his down to earth proof is: "…you try to explain a piece of mathematics to a student or a colleague. You have a long, difficult argument and finally the student understands. What does the student say? The student says, 'I see!' . Summing up the preceding considerations make me remember when I asked Calabi what Geometry was for him: "Some mathematics is Geometry when you can trace finally the origin of the statement, or of its proof, to one of the five senses".

But the most appealing to me is Atiyah's Faustian offer, because it lives in metaphysics. Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, and it will answer any question you like. All you need is to give me your soul: give up geometry and you will have this marvelous machine (algebra or/and a computer). The danger for your soul is there, because when you pass to algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

Very recent, and quite challenging is the "*Dead Sea Discussion"* at the end of the GAFA 2000 (see Bibliography). This discussion is not dead, but quite sharp and provocative, it is fascinating to read, and this for every field. We just extract a Gromov sentence in the geometry section.

Geometry has a structure which is very different from number theory. It just no go a definite way, it is spread. There are particularly difficult questions, some of them are very good and unnatural. We cannot solve them, that's for sure. But there is no one point where it is blocked. It was never like that. Geometry never goes as far. Compared to other branches of mathematics it depends on a different part of your brain. It is not the consecutive part of your brain, exercising long sequences; it is spread like visual perception, so it cannot be blocked. When you see something you cannot be blocked. In geometry you don't go far, ever.

2. The Incredible Successive Enlargements of the Notions of Space and of Point

2.1. Introduction

This is the most important section of our text. New good notions of a "space" were never built up for the sake of generality, but for understanding figures, etc. which were incomprehensible in any of the various kinds of spaces known at the time of the given study. Or, if however partly comprehensible, statements, results, needed constantly exceptions, etc. because we will certainly not be able to present in a detailed enough description all the successive notions of "spaces"; let us quote few historical examples of those approaches (say flows?). The first is the creation of projective space, where two lines always meet: there are no parallels anymore, written $\boldsymbol{R}P^2$ see its definition just below. The second is the introduction of complex numbers in Geometry. Complex numbers were introduced in Algebra in order to have solutions for any quadratic equation, the typical one being $x^2 = -1$; and the geometric extension is to look at two conics in a (real) plane, they can have no common points, or only two, or four (conics are the plane curves defined by a quadratic two variable equation). See the answer in Section 2.2. This is still the realm of classical Euclidian and post-Euclidean geometry, but then in the middle 19$^{\text{th}}$ century came the questioning about the real geometry of the space we are living in. For this long story, still not finished, see the various sections below.

Note that the spaces successively introduced were sometimes more, much more, general than the ones known before. But also some refined, not less general, structures were introduced, see the various sections below. And we end this by quoting Bourbaki's definition of a *set*:

A set is composed of elements capable of having certain properties and having certain relations among themselves or with elements of other sets

Two notions are standard and elementary, *product and quotient*. The product of two spaces X and Y is the new set made up by all the couples (x, y) where x runs through X and y runs through Y. This notion extends to a product of spaces in finite number. But for an infinite number of given sets we enter in the domain of Analysis, convergence of sequences, etc. We mention Hilbert space in Section 2.14, see also Banach spaces in Section 7.

One "sees" product spaces easily, quotient space are notoriously hard to "see". This is making a new space by a partition of subspaces, also called an *equivalence* relation. Difficult is to see the structure of all Penrose tilings in Section 4.13. But a famous historical example is the real projective plane $\boldsymbol{R}P^2$. It cannot be seen smoothly in the ordinary space, this explains why a non-Euclidean geometry (two points give a line, two lines always meet in one point) had to wait so long to be discovered.

2.2. Euclidean, Projective and Complex plane and Space Geometries

At the end of the 19$^{\text{th}}$ century the followings spaces were quite well understood, say precisely defined. Euclidean geometry, plane and space, were correctly defined via algebra, as the sets of all couples or triples of real numbers (of any sign of course), this for the vector space view. The Euclidean structure consisted in adding a distance between points, given explicitly by a quadratic form, say that the distance between (x, y) and (x', y') is by definition the square root of $(x - x')^2 + (y - y')^2$, and the same for the space. Needless to say that one has to make such definitions independent of the choice of coordinate writing. Note that one says "the" because all Euclidean planes, or spaces, are "the same", the technical language is "isomorphic". We here completely ignore all the axiomatic presentations, in particular Hilbert's completely axiomatic definition of the Euclidean plane. There is the field called *Geometric Algebra* (do not confuse with Algebraic Geometry), mainly concerned with axiomatic geometry, we will skip it completely.

The next spaces to have been defined are the projective plane and the projective space. After definitions which were more close to act of faith than a solid setting (and paradoxically called "modern geometry" during most of the XIXth century), the projective plane was defined as the set of all triples written as $(x:y:z)$; this writing meaning that these three real numbers are considered only up to multiplication by a same scalar, or say that two such triples define a point of the projective plane when the two (three) ratios among x, y, z are equal. The result is certainly a "plane", since the dimension left after the quotient is two. The three-dimensional projective space goes the same way, but for quadruples $(x:y:z:t)$.

Back in 1820 Poncelet dared, again as an act of faith, to talk about the complex projective plane without any algebraic setting, Analytic Geometry and coordinates for him were "evil". It is only at the end of the 20th century that one realized that the formal setting was only to consider the triples $(x:y:z)$ of complex numbers (still under the equivalence relation explained above). Historically, Poncelet's work was refused by the Academy of sciences as labeled "romantic" or "four-dimensional" (and in fact this looks like a joke today since things were really four-dimensional when expressed in real numbers).

A basic fact for the projective spaces is their compactness, because to the vector plane or space we precisely add the elements at infinity, then they are under our control, we have "domesticated" the infinite, see more on this in Section 6.3. So we have no exception in the complex projective plane: two conics meet always in four points (with exceptions for contact of higher degree, but then jiggle them a little bit, or talk of the *generic* case, see Section 2.12). Another great discovery of Poncelet was that of the so-called *cyclic points*. They are two points at infinity which bear the *scar* of the Euclidean structure under consideration. Now any circle (understood projectified and complexified) contains the cyclic points, so that any pair of circles has *four* points in common. So it is not surprising that the members of Academy of Sciences in the 1820's were not happy!

2.3. A Dramatic Flow, the Increase in Dimension: Introduce Geometrical Objects of Dimension 4, or more: any Integer *n*, and even more: Introducing Infinite Dimensional-Spaces

Today it seems trivial to define 4 or more n-*dimensional* spaces, just take the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers, and all the linear geometry of that space, denoted by $\boldsymbol{R}^n$, is governed by linear manipulations of the variables considered (called *coordinates*). A Euclidean structure is built now on $\boldsymbol{R}^n$ with the obvious extension of the square root formula given above. And similarly, in any dimension n one can define as above the real projective space $\boldsymbol{RP}^n$ and the complex projective space $\boldsymbol{CP}^n$. It is in those spaces that started the modern algebraic geometry. But up to the 1950s many algebraic geometers were still reasoning “à la Poncelet”; this led to some false and even spectacular statements. Then, solid foundations were laid, starting in the 1950’s,

Most geometers, at the beginning of the 20th century, were still doing geometry in dimensions two or three. This is not to say that there are still many open questions in those primitive dimensions. But the climbing in dimension was a major event of the 20th century. The trivial written dimension above should not hide the fact that, after quite a number of years where for example, four dimensional spaces, were considered as fictitious objects. But let us quote Atiyah: “The idea that you take these things seriously and studied them to an equal degree is really a product of 20th century”.

Of course there were quite a lot of motivations, from Algebra to Analysis: functions of any number of variables, or sets of functions, or for studying vector-valued functions. Think also of describing the solar system with the sets of positions of the sun and the planets. And, very natural to physicists, the set of all positions, and their speed vectors, of sets of N particles, this is a $3N + 3N = 6N$ dimension. Even for the pure geometer such generalizations turned out to be “free”; they permitted him/her to discover completely unexpected phenomena, which were without equivalent in the ordinary space.

Infinite-dimensional spaces come (see Section 7) as spaces of functions. A very particular case is *Hilbert space*, which is the Euclidean space $\boldsymbol{R}^\infty$, more generally Banach spaces (see Section 7) are spaces of functions which are endowed with a norm not necessarily Euclidean. An infinite-dimensional game comes necessarily with Calculus of Variations, more generally Analysis, see Section 3.4, and also Section 7. In all the above examples the wording *dimension* had an obvious meaning, and we did not insist on it. But for more general spaces (and subspaces) “dimension” is a difficult notion, with quite different answers according to the object under study, see Section 3.2.

-
-
-

Bibliography

Alon, N., J. Bourgain, et al. (2000). GAFA 2000 *Visions in Mathematics, towards 2000*, Birkhäuser. [Two edited books, their contributions are exceptionally but nicely provocative and full of insights]

Arnold, V. (2000). *Polymathematics : Is mathematics a single science or a set of arts ? Mathematics: frontiers and perspectives.* A. Arnold, Lax, Mazur, AMS: 403-416. [Provocative typical of Arnold's style, but deep]

Arnold, V. Atiyah, M., Lax,P., Mazur,.B., Eds. (2000). *Mathematics; frontiers and perpsectives*, American math. Soc. and Intern. math. Union.[Edited set of two book, with texts covering at the same time history and perspective, quite in Weil's demand]

Atiyah, M. (2000). *Mathematics in the 20th Century*. World Mathematical Year 2000 Symposium, Toronto.[Is a "tour de force" for its breadth and width. Even if he treats all mathematics and not only geometry, the author does this fulfilling largely Weil's demands (see Section 1.1). And this within 15 pages].

Berger, M. (2003). *A Panoramic View of Riemannian Geometry*, Springer. [A book of 800 pages, up to date in 2003, covering the whole field, the spirit being to give motivations, results and ideas behind the proofs]

Berger, M. (2006). *Convexité dans le plan, dans l'espace et au-delà*, Ellipses. [Two small books, of quite elementary level, but up to date on the field, insisting on motivations, geometric vision and a choice of nice results]

Cartier, P. (1998). La folle journée de Grothendieck à Connes et Konsevich. Les relations entre les mathématiques et la physique théorique, Festschrift for the 40th anniversary of the IHES, IHES, Bures sur Yvette. [Only Cartier could write such a naively provocative, but deep, history including in particular the dramatic evolution of the notions of points and spaces]

Connes, A. (2000). "*NON COMMUTATIVE GEOMETRY*." GAFA 2000, Geometric and Functional Analysis Special volume: 481-559. [By its founder, and still incredibly productive in the field. This text describes all the various parts of mathematics and theoretical physics where non-commutative geometry opened new visions and results]

Connes, A. (2001). *Géométrie non-commutative. Qu'est-ce que l'Univers* ? *Université de tous les savoirs*. Y. Michaud, Odile Jacob. 4: 175-192. [One of the few texts where one can find what is non-commutative geometry without too much mathematical knowledge]

Engguist, B. and W. Schmid, Eds. (2001). *Mathematics unlimited- 2001 and beyond*, Springer. [Edited book with remarkable contributions to both history and perspectives (A treacherous subject)]

Gromov, M. (2000a). *Spaces and questions. Visions in mathematics*, GAFA 2000, towards 2000. B. Alon, Connes, Gromov, Milman, Birkhaüser. I: 118-161. [Extremely original Gromov's visions on various domains in mathematics where vision, random considerations, etc. on today various geometrical structures which enters in apparently completely different domaines]

Hurewicz, W. and H. Wallman (1948). Dimension Theory, Princeton University Press .[The great classic, still pertinent, to the subtle subject of dimension]

Kouneiher, J., Flament (delate the "z"), D., Nabonnand, P., Szczecinarz, J-J, Eds. (2005). *Géométrie au XXe siècle, Histgoire et horizons*, Hermann.

Pier, J.-P., Ed. (2000). Developments of Mathematics 1950-2000, Birkhäuser. [Covering Geometry we have the above two books. They are in some sense at the end of Weil's demands. They cover most of geometrical topics, but in a non unified way for most of their content. One of these two books concerns only half of the century.]

Biographical Sketch

Marcel Berger was born in Paris, France, 1927. After Ph D on "Holonomy groups of Riemannian manifolds" under A. Lichnerowicz, was professor at the Universities of Strasbourg, Nice and Paris 1955-1974. From 1974-1985 and 1994-1997, Director of research with the CNRS (Centre National de la Recherche Scientifique). During 1985-1994, he was Director of the IHES (Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France). Since 1997 director of research emeritus at CNRS. Visited a full year MIT 1956-57 and one full year UC Berkeley 1961-62 plus the summer terms 1968 and 1969. Visiting professor at Osaka University for three months, visiting professor for twice three months at Zürich Polytechnicum, visiting professor for winter term at University of Pennsylvania at Philadelphia (1994), for the spring term at State Univerdsity of New York at Stony Brook (1995).

President of the French Mathematical Society 1979-81. Corresponding member of the French Academy of Sciences since 1982. Rademacher Lecturer (University. of Pennsylvania) 1981. Editor and managing editor of various mathematical journals (*Inventiones mathematicae*, *Geometriae dedicata*, *Compositio mathematica*), and in particular managing editor of *Annales scientifiques de l'École normale supérieure* and of the yellow Springer series *Grundlehren des Mathematischen Wissenschaften.* Also published in the Notices of the AMS and in the Gazette of the French mathematical Society: *Encounter with a geometer I and II (M. Gromov*), in the Notices of the AMS: *What is a systole.*

Published around 45 papers on Riemannian Geometry and the following books: Le spectre d'un variété Riemannienne (Springer), translated in Chinese, Géométrie in two volumes in French, two volumes and II, translated in Russian, and in English (Springer), : (with Gostiaux) :Géométrie différentielle : variétés, courbes et surfaces(PUF), translated in English (Springer), (with Berry, Pansu and St.Raymond) Problems in Geometry (translated form the French in English, also translated into Japanese), A panoramic view of Riemannian Geometry, Springer 2003, Convexité (2 volumes, Ellipses), translated in English (Springer), Géométrie vivante : l'échelle de Jacob(Cassini), in English Modern Geometry (Springer).